Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 227 (2011) 1399–1412
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Smoothness of the moduli space of complexes of coherent sheaves on an abelian or a projective K3 surface

Michi-aki Inaba

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Received 7 May 2010; accepted 7 March 2011
Available online 30 March 2011
Communicated by Bertrand Toen

Abstract

For an abelian or a projective K3 surface X over an algebraically closed field k, consider the moduli space $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ of the objects E in $D^b(\mathrm{Coh}(X))$ satisfying $\mathrm{Ext}^{-1}_X(E, E) = 0$ and $\mathrm{Hom}(E, E) \cong k$. Then we can prove that $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ is smooth and has a symplectic structure.

MSC: 14D20; 18E30

Keywords: Moduli of complexes of coherent sheaves

1. Introduction

It was proved by Mukai in [7] that the moduli space of simple sheaves on an abelian or a projective K3 surface is smooth and has a symplectic structure. We will generalize this result to the moduli space of objects in the derived category of coherent sheaves, which is introduced in [4]. By [5, Theorem 4.4], the moduli space of (semi)stable objects with respect to a strict ample sequence in a derived category of coherent sheaves on an abelian or a projective K3 surface gives examples of projective symplectic varieties.

E-mail address: inaba@math.kyoto-u.ac.jp.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.03.001

In the proof of the main results, we will use the trace map that also played a key role in [7]. More precisely, we will calculate the image by the trace map of the obstruction class for the deformation of complexes of coherent sheaves. So the idea of the proof of this paper is the same as that of [7]. However, the calculation of the trace map, without any preparation, seems to be too complicated. For this reason, we will reconsider in Section 2 the definition of the obstruction class for the deformation of vector bundles. By virtue of this consideration (Lemma 2.3) in Section 2, the calculation of the trace map becomes clear and the main result can be deduced from it.

The content of this paper was originally written as an appendix of [5]. However there was a mistake in the proof of the smoothness of $\mathrm{Splcpx}^{\text{ét}}_{X/k}$. In this paper the author corrects the mistake.

We also prove that the moduli space $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ for X abelian or $K3$ surface has a canonical symplectic structure, which is essentially proved in [2, II-10]. However, our proof is different from that of [2].

2. Obstruction classes for the deformation of vector bundles

First we recall the obstruction theory of the deformation of objects in the derived category of bounded complexes of coherent sheaves.

Let S be a noetherian scheme and X be a projective scheme flat over S. We fix an S-ample line bundle $\mathcal{O}_X(1)$ on X. Let A be a noetherian ring over S and I be an ideal of A such that $I^2 = 0$. Take a bounded complex $E^\bullet$ of A/I-flat coherent sheaves on $X_{A/I}$. Then there are integers l, l' such that $E^i = 0$ for $i < l'$ and $i > l$. We can take a complex $V^\bullet = (V^i, d^i)$ of the form $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ and a quasi-isomorphism $V^\bullet \to E^\bullet$, where V_i are free A modules of finite rank, $V_i = 0$ for $i > l$ and $1 \ll m_l \ll m_{l-1} \ll \cdots \ll m_{i+1} \ll m_i \ll \cdots$. Take lifts

$$\tilde{d}^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$$

of the homomorphisms

$$d^i : V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_{A/I}}(-m_{i+1}).$$

Then we obtain homomorphisms

$$\delta^i := \tilde{d}^{i+1} \circ \tilde{d}^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to I \otimes_A V_{i+2} \otimes \mathcal{O}_{X_A}(-m_{i+2}).$$

We put

$$\omega\big(E^\bullet\big) := \big[\big\{\delta^i\big\}\big] \in H^2\big(\mathrm{Hom}\big(V^\bullet, V^\bullet \otimes I\big)\big) \cong \mathrm{Ext}^2\big(E^\bullet, E^\bullet \otimes^{\mathbf{L}}_{A/I} I\big).$$

Proposition 2.1. *We have $\omega(E^\bullet) = 0$ if and only if $E^\bullet$ can be lifted to an object of $D^b(\mathrm{Coh}(X_A))$ of finite* Tor *dimension over A.*

(Proof is the same as [4, Proposition 2.3].)

For a vector bundle, there is another definition of the obstruction class. Let F be a locally free sheaf of rank r on $X_{A/I}$. Take an affine open covering $\{U_\alpha\}$ of X_A such that $F|_{U_\alpha} \cong \mathcal{O}_{U_\alpha \otimes A/I}^{\oplus r}$ for any α. Let F_α be a free $\mathcal{O}_{U_\alpha}$-module such that $F_\alpha \otimes A/I \cong F|_{U_\alpha}$. Take a lift $\varphi_{\beta\alpha} : F_\alpha|_{U_{\alpha\beta}} \to F_\beta|_{U_{\alpha\beta}}$ of the composite

$$F_\alpha \otimes A/I|_{U_{\alpha\beta}} \xrightarrow{\sim} F|_{U_{\alpha\beta}} \xrightarrow{\sim} F_\beta \otimes A/I|_{U_{\alpha\beta}},$$

where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. We put

$$\theta_{\alpha\beta\gamma} := \varphi_{\gamma\alpha}^{-1} \circ \varphi_{\gamma\beta} \circ \varphi_{\beta\alpha} - \mathrm{id}_{F_\alpha} : F_\alpha|_{U_{\alpha\beta\gamma}} \longrightarrow I \otimes F_\alpha|_{U_{\alpha\beta\gamma}},$$

where $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. Then the cohomology class

$$o(F) := \bigl[\{\theta_{\alpha\beta\gamma}\}\bigr] \in \check{H}^2\bigl(\mathcal{E}nd(F) \otimes I\bigr) \cong \mathrm{Ext}^2(F, F \otimes I)$$

can be defined. As is stated in [1, III, Proposition 7.1], we have the following proposition.

Proposition 2.2. *We have $o(F) = 0$ if and only if F can be lifted to a locally free sheaf on X_A.*

A vector bundle F on $X_{A/I}$ can be considered as the object of $D^b(\mathrm{Coh}(X_{A/I}))$ whose 0-th component is F and the other components are zero. We will show that $\omega(F)$ and $o(F)$ are the same element in $\mathrm{Ext}^2(F, F \otimes I)$.

We take a resolution of F by locally free sheaves:

$$\cdots \longrightarrow V^2 \xrightarrow{d^2} V^1 \xrightarrow{d^1} V^0 \xrightarrow{\pi} F \longrightarrow 0,$$

where each V^i is isomorphic to $V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ for a free A-module V_i of finite rank and $1 \ll m_0 \ll m_1 \ll \cdots \ll m_i \ll m_{i+1} \ll \cdots$. Then we have a quasi-isomorphism $\mathcal{H}om(F, F) \otimes I \to \mathcal{H}om^\bullet(V^\bullet, F) \otimes I$. Let

$$\mathcal{H}om(F, F) \otimes I \to \mathcal{C}^\bullet\bigl(\mathcal{H}om(F, F) \otimes I\bigr)$$

be the Čech resolution of $\mathcal{H}om(F, F) \otimes I$ with respect to the covering $\{U_\alpha\}$ and

$$\mathcal{H}om^\bullet\bigl(V^\bullet, F\bigr) \otimes I \to \mathcal{C}^\bullet\bigl(\mathcal{H}om^\bullet\bigl(V^\bullet, F\bigr) \otimes I\bigr)$$

be that of $\mathcal{H}om^\bullet(V^\bullet, F) \otimes I$. Then we obtain a composition of isomorphisms

$$f : H^2\bigl(\mathrm{Hom}^\bullet\bigl(V^\bullet, F\bigr)\bigr) \xrightarrow{\sim} \mathbf{H}^2\bigl(C^\bullet\bigl(\mathcal{H}om^\bullet\bigl(V^\bullet, F\bigr) \otimes I\bigr)\bigr) \xrightarrow{\sim} \check{H}^2\bigl(\mathcal{E}nd(F) \otimes I\bigr),$$

where $C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I) = \Gamma(X, \mathcal{C}^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I))$.

Lemma 2.3. *Under the above assumption and notation, we have $f(\omega(F)) = o(F)$.*

Proof. First note that the element $\omega(F)$ is defined by

$$\omega(F) = \big[\big\{(\pi \otimes \mathrm{id}_I) \circ \big(\tilde{d}^1 \circ \tilde{d}^2\big)\big\}\big] \in H^2\big(\mathrm{Hom}^\bullet\big(V^\bullet, F \otimes I\big)\big),$$

where $\tilde{d}^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$ is a lift of d^i. Replacing $\{U_\alpha\}$ by its refinement, we may assume that $\ker d^2|_{U_\alpha}$, $\mathrm{im}\, d^2|_{U_\alpha}$, $\mathrm{im}\, d^1|_{U_\alpha}$, $V_2 \otimes \mathcal{O}_{X_{A/I}}(-m_2)|_{U_\alpha}$, $V_1 \otimes \mathcal{O}_{X_{A/I}}(-m_1)|_{U_\alpha}$ and $F|_{U_\alpha}$ are all free sheaves. Then the exact sequences

$$0 \longrightarrow \ker d^2\big|_{U_\alpha} \xrightarrow{i_2} V_2 \otimes \mathcal{O}_{X_{A/I}}(-m_2)|_{U_\alpha} \xrightarrow{p_2} \mathrm{im}\, d^2\big|_{U_\alpha} \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{im}\, d^2\big|_{U_\alpha} \xrightarrow{i_1} V_1 \otimes \mathcal{O}_{X_{A/I}}(-m_1)|_{U_\alpha} \xrightarrow{p_1} \mathrm{im}\, d^1\big|_{U_\alpha} \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{im}\, d^1\big|_{U_\alpha} \xrightarrow{i_0} V_0 \otimes \mathcal{O}_{X_{A/I}}(-m_0)|_{U_\alpha} \xrightarrow{\pi|_{U_\alpha}} F|_{U_\alpha} \longrightarrow 0$$

split and we can take free $\mathcal{O}_{U_\alpha}$-modules F_α, I_1^α, I_2^α such that $F_\alpha \otimes A/I \cong F|_{U_\alpha}$ and $I_i^\alpha \otimes A/I \cong \mathrm{im}\, d^i|_{U_\alpha}$ for $i = 1, 2$. Taking lifts $\tilde{i}_0^\alpha$, $\tilde{i}_1^\alpha$, π_α, $\tilde{p}_1^\alpha$, $\tilde{p}_2^\alpha$ of i_0, i_1, $\pi|_{U_\alpha}$, p_1, p_2, we obtain splitting exact sequences

$$0 \longrightarrow \ker \tilde{p}_2^\alpha \longrightarrow V_2 \otimes \mathcal{O}_{X_A}(-m_2)|_{U_\alpha} \xrightarrow{\tilde{p}_2^\alpha} I_2^\alpha \longrightarrow 0,$$
$$0 \longrightarrow I_2^\alpha \xrightarrow{\tilde{i}_1^\alpha} V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha} \xrightarrow{\tilde{p}_1^\alpha} I_1^\alpha \longrightarrow 0,$$
$$0 \longrightarrow I_1^\alpha \xrightarrow{\tilde{i}_0^\alpha} V_0 \otimes \mathcal{O}_{X_A}(-m_0)|_{U_\alpha} \xrightarrow{\pi_\alpha} F_\alpha \longrightarrow 0.$$

Let

$$\tilde{s}_2^\alpha : I_2^\alpha \longrightarrow V_2 \otimes \mathcal{O}_{X_A}(-m_2)|_{U_\alpha},$$
$$\tilde{r}_1^\alpha : V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha} \longrightarrow I_2^\alpha, \quad \tilde{s}_1^\alpha : I_1^\alpha \longrightarrow V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha},$$
$$\tilde{r}_0^\alpha : V_0 \otimes \mathcal{O}_{X_A}(-m_0)|_{U_\alpha} \longrightarrow I_1^\alpha, \quad \nu_\alpha : F_\alpha \longrightarrow V_0 \otimes \mathcal{O}_{X_A}(-m_0)|_{U_\alpha}$$

be splittings. Put

$$d_\alpha^2 : V_2 \otimes \mathcal{O}_{X_A}(-m_2)|_{U_\alpha} \xrightarrow{\tilde{p}_2^\alpha} I_2^\alpha \xrightarrow{\tilde{i}_1^\alpha} V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha},$$
$$d_\alpha^1 : V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha} \xrightarrow{\tilde{p}_1^\alpha} I_1^\alpha \xrightarrow{\tilde{i}_0^\alpha} V_0 \otimes \mathcal{O}_{X_A}(-m_0)|_{U_\alpha},$$
$$\tau_\alpha : V_0 \otimes \mathcal{O}_{X_A}(-m_0)|_{U_\alpha} \xrightarrow{\tilde{r}_0^\alpha} I_1^\alpha \xrightarrow{\tilde{s}_1^\alpha} V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha},$$
$$\sigma_\alpha : V_1 \otimes \mathcal{O}_{X_A}(-m_1)|_{U_\alpha} \xrightarrow{\tilde{r}_1^\alpha} I_2^\alpha \xrightarrow{\tilde{s}_2^\alpha} V_2 \otimes \mathcal{O}_{X_A}(-m_2)|_{U_\alpha}.$$

We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(V^0, F\otimes I) & \longrightarrow & \mathrm{Hom}(V^1, F\otimes I) & \longrightarrow & \mathrm{Hom}(V^2, F\otimes I) \\
\downarrow & & \downarrow & & \downarrow \\
C^0(\mathcal{H}om(V^0, F\otimes I)) & \longrightarrow & C^0(\mathcal{H}om(V^1, F\otimes I)) & \longrightarrow & C^0(\mathcal{H}om(V^2, F\otimes I)) \\
\downarrow & & \downarrow & & \downarrow \\
C^1(\mathcal{H}om(V^0, F\otimes I)) & \longrightarrow & C^1(\mathcal{H}om(V^1, F\otimes I)) & \longrightarrow & C^1(\mathcal{H}om(V^2, F\otimes I)) \\
\downarrow & & \downarrow & & \downarrow \\
C^2(\mathcal{H}om(V^0, F\otimes I)) & \longrightarrow & C^2(\mathcal{H}om(V^1, F\otimes I)) & \longrightarrow & C^2(\mathcal{H}om(V^2, F\otimes I)),
\end{array}$$

where we put $V^i := V_i \otimes \mathcal{O}_{X_A}(-m_i)$ for $i = 0,1,2$. The image of $\omega(F)$ in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F)\otimes I))$ can be represented by

$$\big\{(\pi\otimes \mathrm{id}_I)\circ \tilde{d}^1\circ \tilde{d}^2\big|_{U_\alpha}\big\} \in C^0\big(\mathcal{H}om\big(V^2, F\big)\otimes I\big),$$

which defines the same element in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F)\otimes I))$ as

$$\big\{(\pi\otimes \mathrm{id}_I)\circ \tilde{d}^1\circ \tilde{d}^2\circ(\sigma_\alpha-\sigma_\beta)\big\} \in C^1\big(\mathcal{H}om\big(V^1, F\big)\otimes I\big).$$

On the other hand, the image of the element

$$\big\{(\pi\otimes \mathrm{id}_I)\circ\big(d^1_\alpha - \tilde{d}^1\circ\big(1-\tilde{d}^2\sigma_\alpha\big)\big)\big\} \in C^0\big(\mathcal{H}om\big(V^1, F\big)\otimes I\big)$$

by the homomorphism $C^0(\mathcal{H}om(V^1, F)\otimes I) \to C^0(\mathcal{H}om(V^2, F)\otimes I)$ is

$$\begin{aligned}
&\big\{(\pi\otimes \mathrm{id}_I)\circ\big(d^1_\alpha - \tilde{d}^1\circ\big(1-\tilde{d}^2\circ\sigma_\alpha\big)\big)\circ d^2_\alpha\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\big(d^1_\alpha\circ d^2_\alpha - \tilde{d}^1\circ d^2_\alpha + \tilde{d}^1\circ\tilde{d}^2\circ\sigma_\alpha\circ d^2_\alpha\big)\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\big(-\tilde{d}^1\circ d^2_\alpha + \tilde{d}^1\circ\tilde{d}^2\circ\sigma_\alpha\circ d^2_\alpha\big)\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\big(-\tilde{d}^1\circ d^2_\alpha + \tilde{d}^1\circ\tilde{d}^2\circ\tilde{s}^\alpha_2\circ\tilde{r}^\alpha_1\circ\tilde{i}^\alpha_1\circ\tilde{p}^\alpha_2\big)\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\big(-\tilde{d}^1\circ d^2_\alpha\circ\tilde{s}^\alpha_2\circ\tilde{p}^\alpha_2 + \tilde{d}^1\circ\tilde{d}^2\circ\tilde{s}^\alpha_2\circ\tilde{p}^\alpha_2\big)\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\circ\tilde{d}^1\circ\big(\tilde{d}^2 - d^2_\alpha\big)\circ\tilde{s}^\alpha_2\circ\tilde{p}^\alpha_2\big\} \\
&\quad = 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\big\{(\pi\otimes \mathrm{id}_I)\circ\tilde{d}^1\circ\tilde{d}^2\circ(\sigma_\alpha-\sigma_\beta)\big\} + d\big\{(\pi\otimes \mathrm{id}_I)\circ\big(d^1_\alpha - \tilde{d}^1\circ\big(1-\tilde{d}^2\circ\sigma_\alpha\big)\big)\big\} \\
&\quad = \big\{(\pi\otimes \mathrm{id}_I)\circ\tilde{d}^1\circ\tilde{d}^2\circ(\sigma_\alpha-\sigma_\beta)\big\} + \big\{(\pi\otimes \mathrm{id}_I)\circ\big(d^1_\beta - \tilde{d}^1\circ\big(1-\tilde{d}^2\circ\sigma_\beta\big)\big)\big|_{U_\alpha\cap U_\beta}\big\}
\end{aligned}$$

$$
\begin{aligned}
&- \bigl\{(\pi \otimes \mathrm{id}_I) \circ \bigl(d^1_\alpha - \tilde{d}^1 \circ \bigl(1 - \tilde{d}^2 \circ \sigma_\alpha\bigr)\bigr)\big|_{U_\alpha \cap U_\beta}\bigr\} \\
&= -\bigl\{(\pi \otimes \mathrm{id}_I) \circ \bigl(d^1_\alpha - d^1_\beta\bigr)\bigr\},
\end{aligned}
$$

we can see that $\{(\pi \otimes \mathrm{id}_I) \circ \tilde{d}^1 \circ \tilde{d}^2(\sigma_\alpha - \sigma_\beta)\}$ and $-\{(\pi \otimes \mathrm{id}_I) \circ (d^1_\alpha - d^1_\beta)\}$ define the same element in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I))$. We can see that the element $-\{(\pi \otimes \mathrm{id}_I) \circ (d^1_\alpha - d^1_\beta)\}$ defines the same element as

$$
\begin{aligned}
&-\bigl\{(\pi \otimes \mathrm{id}_I) \circ \bigl(\bigl(d^1_\beta - d^1_\gamma\bigr) \circ \tau_\beta - \bigl(d^1_\alpha - d^1_\gamma\bigr) \circ \tau_\alpha + \bigl(d^1_\alpha - d^1_\beta\bigr) \circ \tau_\alpha\bigr)\bigr\} \\
&\quad = \bigl\{(\pi \otimes \mathrm{id}_I) \circ \bigl(d^1_\beta - d^1_\gamma\bigr) \circ (\tau_\alpha - \tau_\beta)\bigr\} \in C^2\bigl(\mathcal{H}om\bigl(V^0, F\bigr) \otimes I\bigr)
\end{aligned}
$$

in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I))$. Thus $\omega(F)$ is equal to the element given by

$$
\bigl\{(\pi \otimes \mathrm{id}_I) \circ \bigl(d^1_\beta - d^1_\gamma\bigr) \circ (\tau_\alpha - \tau_\beta)\bigr\} \in C^2\bigl(\mathcal{H}om\bigl(V^0, F\bigr) \otimes I\bigr)
$$

in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I))$. On the other hand, the element $o(F)$ is given by

$$
\bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha - \mathrm{id}_{F_\alpha}\bigr\}
$$

in $\check{H}^2(\mathcal{E}nd(F) \otimes I)$, whose image in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F) \otimes I))$ is represented by

$$
\begin{aligned}
&\bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha \circ \pi_\alpha - \pi_\alpha\bigr\} \\
&\quad = \bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ (\pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha \circ \pi_\alpha - \pi_\gamma \circ \nu_\alpha \circ \pi_\alpha)\bigr\} \\
&\quad = \bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ (\nu_\beta \circ \pi_\beta - 1) \circ \nu_\alpha \circ \pi_\alpha\bigr\} \\
&\quad = \bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ \bigl(-d^1_\beta \circ \tau_\beta\bigr) \circ \bigl(1 - d^1_\alpha \circ \tau_\alpha\bigr)\bigr\} \\
&\quad = \bigl\{(\pi_\gamma \circ \nu_\alpha)^{-1} \circ \bigl(\pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) - \pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) \circ d^1_\alpha \circ \tau_\alpha\bigr)\bigr\}.
\end{aligned}
$$

Here we have

$$
\begin{aligned}
&\pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) \circ d^1_\alpha \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \bigl(\tilde{s}^\alpha_1 \circ \tilde{r}^\alpha_0 - \tilde{s}^\beta_1 \circ \tilde{r}^\beta_0\bigr) \circ d^1_\alpha \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\alpha_1 \circ \tilde{r}^\alpha_0 \circ d^1_\alpha \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\beta_1 \circ \tilde{r}^\beta_0 \circ \bigl(d^1_\alpha - d^1_\beta\bigr) \circ \tau_\alpha \\
&\qquad - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\beta_1 \circ \tilde{r}^\beta_0 \circ d^1_\beta \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\alpha_1 \circ \tilde{r}^\alpha_0 \circ \tilde{i}^\alpha_0 \circ \tilde{p}^\alpha_1 \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\beta_1 \circ \tilde{r}^\beta_0 \circ \tilde{i}^\beta_0 \circ \tilde{p}^\beta_1 \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\alpha_1 \circ \tilde{p}^\alpha_1 \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^\beta_1 \circ \tilde{p}^\beta_1 \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \bigl(\mathrm{id} - \tilde{i}^\alpha_1 \circ \tilde{r}^\alpha_1\bigr) \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \bigl(\mathrm{id} - \tilde{i}^\beta_1 \circ \tilde{r}^\beta_1\bigr) \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \tilde{i}^\alpha_1 \circ \tilde{r}^\alpha_1 \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{i}^\beta_1 \circ \tilde{r}^\beta_1 \circ \tau_\alpha \\
&\quad = \pi_\gamma \circ d^1_\beta \circ \tilde{i}^\alpha_1 \circ \tilde{r}^\alpha_1 \circ \tilde{s}^\alpha_1 \circ \tilde{r}^\alpha_0 \quad \bigl(\text{note that } d^1_\beta \circ \tilde{i}^\beta_1 = 0\bigr) \\
&\quad = 0 \quad \bigl(\text{note that } \tilde{r}^\alpha_1 \circ \tilde{s}^\alpha_1 = 0\bigr).
\end{aligned}
$$

So the image of $o(F)$ in $\mathbf{H}^2(C^\bullet(\mathcal{H}om^\bullet(V^\bullet, F)\otimes I))$ is

$$\begin{aligned}\{(\pi_\gamma\circ\nu_\alpha)^{-1}\circ\pi_\gamma\circ d^1_\beta\circ(\tau_\alpha-\tau_\beta)\} &= \{(\pi_\gamma\circ\nu_\alpha)^{-1}\circ\pi_\gamma\circ\bigl(d^1_\beta-d^1_\gamma\bigr)\circ(\tau_\alpha-\tau_\beta)\}\\ &= \{(\pi\otimes\mathrm{id}_I)\circ\bigl(d^1_\beta-d^1_\gamma\bigr)\circ(\tau_\alpha-\tau_\beta)\}.\end{aligned}$$

Thus we have the equality $f(\omega(F)) = o(F)$. □

Remark 2.4. Several authors introduced obstruction classes for the deformation of vector bundles and coherent sheaves. For example, [2, Chapter 2, Appendix] is a good reference. We should also remark that a most generalized obstruction class is defined in [3] as the product of the Atiyah class and Kodaira–Spencer class. However, it is not so clear that these definitions are all equivalent.

3. Smoothness and symplectic structure

Let X be a projective scheme over a noetherian scheme S, which is flat over S. We define a functor $\mathrm{Splcpx}_{X/S}$ of the category of locally noetherian schemes to that of sets by putting

$$\mathrm{Splcpx}_{X/S}(T) := \left\{ E^\bullet \;\middle|\; \begin{array}{c} E^\bullet \text{ is a bounded complex of } T\text{-flat coherent}\\ \mathcal{O}_{X_T}\text{-modules such that for any } t\in T,\\ E^\bullet(t) \text{ satisfies the following condition } (*)\end{array} \right\} \Big/ \sim,$$

where T is a locally noetherian scheme over S and $E^\bullet \sim F^\bullet$ if there is a line bundle L on T such that $E^\bullet \cong F^\bullet \otimes L$ in $D(X_T)$. Here $D(X_T)$ is the derived category of $\mathcal{O}_{X_T}$-modules and the condition $(*)$ is

$$\mathrm{Ext}^i\bigl(E^\bullet(t), E^\bullet(t)\bigr) \cong \begin{cases} 0 & \text{if } i=-1,\\ k(t) & \text{if } i=0.\end{cases} \tag{$*$}$$

Note that we denote $E^\bullet \otimes^{\mathbf{L}} k(t)$ by $E^\bullet(t)$. Let $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ be the étale sheafification of $\mathrm{Splcpx}_{X/S}$.

Theorem 3.1. *The moduli functor* $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ *is represented by an algebraic space locally of finite type over* S.

(Proof is in [4, Theorem 0.2]. This result was generalized by Lieblich in [6] for X proper over S.)

Theorem 3.2. *If X is an abelian or a projective K3 surface over an algebraically closed field k,* $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ *is smooth over k.*

Proof. Take an artinian local ring A over k with residue field $k = A/m$ and an ideal I of A such that $mI = 0$. It is sufficient to show that $\mathrm{Splcpx}_{X/k}(A) \to \mathrm{Splcpx}_{X/k}(A/I)$ is surjective. Indeed we can take a scheme U locally of finite type over k and a morphism $p: U \to \mathrm{Splcpx}_{X/k}$ such that the composite $U \xrightarrow{p} \mathrm{Splcpx}_{X/k} \xrightarrow{\iota} \mathrm{Splcpx}^{\text{ét}}_{X/k}$ is étale and surjective. Take any artinian local ring A over k with residue field $k = A/m$ and an ideal I of A such that $mI = 0$. Take any member $x \in U(A/I)$. By the surjectivity of $\mathrm{Splcpx}_{X/k}(A) \to \mathrm{Splcpx}_{X/k}(A/I)$, we can take an element $y \in \mathrm{Splcpx}_{X/k}(A)$ such that $y \otimes A/I = p(x)$. Then $\iota(y) \in \mathrm{Splcpx}^{\text{ét}}_{X/k}(A)$ and $\iota(y) \otimes A/I = (\iota \circ p)(x)$. Since $\iota \circ p : U \to \mathrm{Splcpx}^{\text{ét}}_{X/k}$ is étale, there is an element $z \in U(A)$ such that $z \otimes A/I = x$ and $(\iota \circ p)(z) = y$. Thus U is smooth over k.

Let $E^\bullet$ be an A/I-valued point of $\mathrm{Splcpx}_{X/k}$. Put $E^\bullet_0 := E^\bullet \otimes k$ and

$$l' := \min\bigl\{i \bigm| H^i\bigl(E^\bullet_0 \otimes^{\mathbf{L}} k(x)\bigr) \neq 0 \text{ for some } x \in X\bigr\}.$$

We may assume that $E^\bullet$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow E^{l'} \xrightarrow{d^{l'}_{E^\bullet}} V^{l'+1} \xrightarrow{d^{l'+1}} \cdots \longrightarrow V^l \xrightarrow{d^l} 0 \longrightarrow 0 \cdots,$$

where $E^{l'}$ is a vector bundle on $X_{A/I}$, $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ with V_i a finite dimensional vector space over k, $\mathcal{O}_X(1)$ a fixed ample line bundle on X and $1 \ll m_l \ll m_{l-1} \ll \cdots \ll m_{l'+1}$. We can see that $d^{l'}_{E^\bullet_0} \otimes k(x)$ is not injective for some $x \in X$. Take a resolution

$$\cdots \longrightarrow V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i) \longrightarrow \cdots \longrightarrow V_{l'} \otimes \mathcal{O}_{X_{A/I}}(-m_{l'}) \xrightarrow{\pi} E^{l'} \longrightarrow 0,$$

where each V_i is a vector space over k of finite dimension and

$$m_{l'+1} \ll m_{l'} \ll \cdots \ll m_i \ll m_{i-1} \ll \cdots.$$

We put $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ for $i \leqslant l$ and $V^i = 0$ for $i > l$. Let $V^\bullet$ be the complex

$$\cdots \longrightarrow V^i \longrightarrow V^{i+1} \longrightarrow \cdots \longrightarrow V^{l'} \xrightarrow{d^{l'}_{E^\bullet} \circ \pi} V^{l'+1} \longrightarrow \cdots \longrightarrow V^l \longrightarrow 0 \longrightarrow \cdots.$$

Then there is a canonical quasi-isomorphism

$$V^\bullet \longrightarrow E^\bullet.$$

Put $V^\bullet_0 := V^\bullet \otimes k$. Let

$$\mathrm{tr}^\bullet : \mathcal{H}om^\bullet\bigl(E^\bullet_0, E^\bullet_0\bigr) \xrightarrow{\sim} \mathcal{H}om^\bullet\bigl(\mathcal{H}om^\bullet\bigl(E^\bullet_0, E^\bullet_0\bigr), \mathcal{O}_X\bigr) \longrightarrow \mathcal{O}_X$$

be the dual of the canonical morphism

$$\mathcal{O}_X \longrightarrow \mathcal{H}om^\bullet\bigl(E^\bullet_0, E^\bullet_0\bigr); \qquad 1 \mapsto \mathrm{id}_{E^\bullet_0}.$$

Note that $\mathrm{tr}^p = 0$ on $\mathcal{H}om^p(E^\bullet_0, E^\bullet_0)$ for $p \neq 0$ and $\mathrm{tr}^0(\{x^i\}) = \sum_i (-1)^i \mathrm{tr}(x^i)$ for $x^i \in \mathcal{H}om(E^i_0, E^i_0)$. We remark that $\mathrm{tr}^\bullet$ is also introduced in [2, Chapter 10]. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^2_X(E^\bullet_0, E^\bullet_0) \arrow[r, "H^2(\mathrm{tr}^\bullet)"] \arrow[d, "s_1"', "\cong"] & H^2(X, \mathcal{O}_X) \arrow[d, "s_2"', "\cong"] \\
\mathrm{Hom}_{D(X)}(E^\bullet_0, E^\bullet_0)^\vee \arrow[r] & H^0(X, \mathcal{O}_X)^\vee,
\end{tikzcd}$$

where s_1, s_2 are the isomorphisms determined by Grothendieck–Serre duality and the bottom row is the dual of $k = H^0(\mathcal{O}_X) \to \operatorname{Hom}_{D(X)}(E_0^\bullet, E_0^\bullet)$, which is bijective since $E_0^\bullet$ is simple. Thus the homomorphism

$$\operatorname{Ext}^2_X\bigl(E_0^\bullet, E_0^\bullet\bigr) \xrightarrow{H^2(\mathrm{tr}^\bullet)} H^2(X, \mathcal{O}_X)$$

is an isomorphism.

Note that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{H}om^\bullet(E^{l'}[-l'], I \otimes E^\bullet) \arrow[r] \arrow[d] & \mathcal{H}om(E^{l'}, I \otimes E^{l'}) \arrow[d, "(-1)^{l'}\mathrm{tr}"] \\
\mathcal{H}om^\bullet(E^\bullet, I \otimes E^\bullet) \arrow[r, "\mathrm{tr}"] & \mathcal{O}_X \otimes I.
\end{tikzcd}$$

From the above commutative diagram, we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^2(E^{l'}[-l'], I \otimes E^\bullet) \arrow[r, "\tau"] \arrow[d, "\sigma"'] & \operatorname{Ext}^2(E^{l'}, I \otimes E^{l'}) \arrow[d, "(-1)^{l'}H^2(\mathrm{tr})"'] \\
\operatorname{Ext}^2(E^\bullet, I \otimes E^\bullet) \arrow[r, "H^2(\mathrm{tr})"] & H^2(\mathcal{O}_X) \otimes I.
\end{tikzcd} \tag{††}$$

Note that the morphism

$$\operatorname{Hom}\bigl(E_0^\bullet, E_0^\bullet\bigr) \longrightarrow \operatorname{Hom}\bigl(E_0^\bullet, E_0^{l'}\bigl[-l'\bigr]\bigr)$$

is not zero, since the image of id by this morphism is the canonical morphism $\iota : E_0^\bullet \to E_0^{l'}[-l']$ which is not zero because

$$H^{l'}\bigl(\iota \otimes k(x)\bigr) : \ker\bigl(d^{l'}_{E_0^\bullet} \otimes k(x)\bigr) = H^{l'}\bigl(E_0^\bullet \otimes k(x)\bigr) \longrightarrow H^l\bigl(E_0^{l'}\bigl[-l'\bigr] \otimes k(x)\bigr) = E_0^{l'} \otimes k(x)$$

is not zero. By Grothendieck–Serre duality, we can see that

$$\iota^* : \operatorname{Ext}^2\bigl(E_0^{l'}\bigl[-l'\bigr], E_0^\bullet\bigr) \longrightarrow \operatorname{Ext}^2\bigl(E_0^\bullet, E_0^\bullet\bigr)$$

is not zero. Since $\operatorname{Ext}^2(E_0^\bullet, E_0^\bullet) \cong k$, ι^* is surjective. So the morphism

$$\sigma : \operatorname{Ext}^2\bigl(E^{l'}\bigl[-l'\bigr], I \otimes E^\bullet\bigr) \longrightarrow \operatorname{Ext}^2\bigl(E^\bullet, I \otimes E^\bullet\bigr)$$

is also surjective.

Take an obstruction class $\omega(E^\bullet) \in \operatorname{Ext}^2(E^\bullet, I \otimes E^\bullet)$ for the lifting of $E^\bullet$ to an A-valued point of $\mathrm{Splcpx}_{X/k}$. Then there is a member $\varphi = [(\varphi^i)] \in \operatorname{Ext}^2(E^{l'}[-l'], I \otimes E^\bullet)$ such that $\sigma(\varphi) = \omega(E^\bullet)$. Here $\varphi^i : V^i \to I \otimes E^{i+2}$ $(i \leqslant l')$ and $\varphi^i = 0$ for $i > l'$. There is an element $\gamma = (\gamma^i) \in \operatorname{Hom}^1(V^\bullet, I \otimes E^\bullet)$ such that

$$\gamma^{i+1} \circ d^i_{V^\bullet} + d^{i+1}_{E^\bullet} \circ \gamma^i = \tilde{d}^{i+1} \circ \tilde{d}^i - \varphi^i \quad \bigl(\text{for } i \geqslant l'-1\bigr),$$
$$\gamma^{l'-1} \circ d^{l'-2}_{V^\bullet} = \pi \circ \tilde{d}^{l'-1} \circ \tilde{d}^{l'-2} - \varphi^{l'-2},$$

where $\tilde{d}^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$ is a lift of $d^i_{V^\bullet}$. We can see that the image of φ by the morphism $\tau : \mathrm{Ext}^2(E^{l'}[-l'], I \otimes E^\bullet) \to \mathrm{Ext}^2(E^{l'}, I \otimes E^{l'})$ is given by $[\pi \circ \tilde{d}^{l'-1} \circ \tilde{d}^{l'-2}]$, which is just the obstruction class $\omega(E^{l'})$. By Lemma 2.3, we have $\omega(E^{l'}) = o(E^{l'})$. We can see that $H^2(\mathrm{tr})(o(E^{l'})) = o(\det(E^{l'}))$. Since the Picard scheme $\mathrm{Pic}_{X/k}$ is smooth over k, we have $o(\det(E^{l'})) = 0$. So we have

$$\begin{aligned} H^2(\mathrm{tr})\bigl(\omega\bigl(E^\bullet\bigr)\bigr) &= H^2(\mathrm{tr})\bigl(\sigma(\varphi)\bigr) \\ &= (-1)^{l'} H^2(\mathrm{tr})\bigl(\tau(\varphi)\bigr) \\ &= (-1)^{l'} H^2(\mathrm{tr})\bigl(\omega\bigl(E^{l'}\bigr)\bigr) \\ &= (-1)^{l'} H^2(\mathrm{tr})\bigl(o\bigl(\bigl(E^{l'}\bigr)\bigr)\bigr) \\ &= (-1)^{l'} o\bigl(\det\bigl(E'\bigr)\bigr) = 0. \end{aligned}$$

Since the morphism

$$H^2(\mathrm{tr}) : \mathrm{Ext}^2(E, I \otimes E) \longrightarrow H^2(\mathcal{O}_X) \otimes I$$

is isomorphic, we have $\omega(E^\bullet) = 0$. Thus $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ is smooth over k. □

The following theorem is essentially proved in [2, II-10]. We give a proof again.

Theorem 3.3. *Let X be an abelian or a projective K3 surface over an algebraically closed field k. Then $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ has a symplectic structure, that is, there exists a closed 2-form on $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ which is nondegenerate at every point.*

Proof. Note that the tangent bundle $T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}}$ on $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ can be considered as the sheaf on the small étale site on $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ defined by

$$U \mapsto \left\{ v \in \mathrm{Splcpx}^{\text{ét}}_{X/k}(U_{k[\epsilon]}) \;\middle|\; \begin{array}{l} \text{the composite } U \xrightarrow{i_0} U_{k[\epsilon]} \xrightarrow{v} \mathrm{Splcpx}^{\text{ét}}_{X/k} \\ \text{is the structure morphism } U \to \mathrm{Splcpx}^{\text{ét}}_{X/k} \end{array} \right\},$$

for any algebraic space U étale over $\mathrm{Splcpx}^{\text{ét}}_{X/k}$, where $k[\epsilon]$ is the k-algebra generated by ϵ with $\epsilon^2 = 0$ and $U \xrightarrow{i_0} U_{k[\epsilon]}$ is the morphism induced by the ring homomorphism

$$k[\epsilon] \longrightarrow k; \quad \epsilon \mapsto 0.$$

There is an étale covering $\coprod_i U_i \to \mathrm{Splcpx}^{\text{ét}}_{X/k}$ such that $U_i \to \mathrm{Splcpx}^{\text{ét}}_{X/k}$ factors through $\mathrm{Splcpx}_{X/k}$, that is, there is a universal family $E^\bullet_{U_i}$ on each X_{U_i}. Let U be an affine scheme étale over $\coprod_i U_i$ and $E^\bullet_U$ be the pull-back of the universal family. Take any element $v \in T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}}(U)$.

Then $U_{k[\epsilon]} \xrightarrow{v} \mathrm{Splcpx}^{\text{ét}}_{X/k}$ factors through $\coprod_i U_i$, since $\coprod_i U_i$ is étale over $\mathrm{Splcpx}^{\text{ét}}_{X/k}$. Let $E^\bullet_{U_{k[\epsilon]}} \in \mathrm{Splcpx}_{X/k}(U_{k[\epsilon]})$ be the pull-back of the universal family. We can take a complex $\tilde{V}^\bullet$ of the form $\tilde{V}^i = V_i \otimes_{\mathcal{O}_U} \mathcal{O}_{X_{U_{k[\epsilon]}}}(-m_i)$ and a quasi-isomorphism $\tilde{V}^\bullet \to E^\bullet_{U_{k[\epsilon]}}$, where V_i is a locally free sheaf of finite rank on U, $V_i = 0$ for $i \gg 0$, $\mathcal{O}_X(1)$ is a fixed ample line bundle on X and $\cdots \gg m_i \gg m_{i+1} \gg \cdots$. Let $V^\bullet$ be the pull-back of $\tilde{V}^\bullet$ by $X \times U \xrightarrow{\mathrm{id}_X \times i_0} X \times U_{k[\epsilon]}$. Then we obtain an element

$$\left[\left\{d^i_{\tilde{V}^\bullet} - d^i_{V^\bullet} \otimes 1\right\}\right] \in H^1\left(\mathrm{Hom}\left(\tilde{V}^\bullet, \epsilon k[\epsilon] \otimes \tilde{V}^\bullet\right)\right) \cong \mathrm{Ext}^1\left(E^\bullet_U, E^\bullet_U\right),$$

which is independent of the choice of the representative $\tilde{V}^\bullet$. We can see that the mapping $v \to [\{d^i_{\tilde{V}^\bullet} - d^i_{V^\bullet} \otimes 1\}]$ defines an isomorphism

$$T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}}(U) \xrightarrow{\sim} H^0\left(U, \mathrm{Ext}^1_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right)\right).$$

For an affine scheme U étale over $\coprod_i U_i$, there is a canonical pairing:

$$\begin{array}{rcl} \alpha_U : \mathrm{Ext}^1_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right) \times \mathrm{Ext}^1_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right) & \longrightarrow & \mathrm{Ext}^2_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right) \\ (g,h) & \mapsto & g \circ h. \end{array}$$

Note that there are canonical isomorphisms

$$\mathrm{Ext}^2_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right) \xrightarrow{\sim} \mathrm{Ext}^0_{X_U/U}\left(E^\bullet_U, E^\bullet_U\right)^\vee \xrightarrow{\sim} \mathcal{O}_U.$$

Consider the projections

$$p_1, p_2 : U \times_{\mathrm{Splcpx}^{\text{ét}}_{X/k}} U \longrightarrow U.$$

For any affine open subscheme $W \subset U \times_{\mathrm{Splcpx}^{\text{ét}}_{X/k}} U$, there are a line bundle $\mathcal{L}$ on W and an isomorphism $p_1^*(E^\bullet_U)_W \cong p_2^*(E^\bullet_U)_W \otimes \mathcal{L}$ in $D^b(\mathrm{Coh}(X \times W))$. So we can easily see that $p_1^*(\alpha_U) = p_2^*(\alpha_U)$ and we can obtain a pairing

$$\alpha : T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}} \times T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}} \longrightarrow \mathcal{O}_{\mathrm{Splcpx}^{\text{ét}}_{X/k}}$$

by patching α_U.

Now we will see that α is skew-symmetric. Take any k-valued point p of $\mathrm{Splcpx}^{\text{ét}}_{X/k}$. p corresponds to a complex

$$\cdots \longrightarrow V_i \otimes \mathcal{O}_X(-m_i) \xrightarrow{d^i_{V^\bullet}} V_{i+1} \otimes \mathcal{O}_X(-m_{i+1}) \xrightarrow{d^{i+1}_{V^\bullet}} \cdots \longrightarrow V_l \otimes \mathcal{O}_X(-m_l) \longrightarrow 0 \longrightarrow \cdots.$$

We denote this complex by $V^\bullet$. Let us consider the restriction

$$\alpha(p) : \mathrm{Ext}^1\left(V^\bullet, V^\bullet\right) \times \mathrm{Ext}^1\left(V^\bullet, V^\bullet\right) \longrightarrow \mathrm{Ext}^2\left(V^\bullet, V^\bullet\right) \cong k$$

of the pairing α. Take any element $v=[\{v^i\}]\in H^1(\mathrm{Hom}^\bullet(V^\bullet,V^\bullet))\cong \mathrm{Ext}^1(V^\bullet,V^\bullet)$ and let $\tilde{V}^\bullet$ be a member of $\mathrm{Splcpx}_{X/k}(k[\epsilon])$ which corresponds to v. $\tilde{V}^\bullet$ can be given by the complex

$$\cdots\longrightarrow V_i\otimes\mathcal{O}_X(-m_i)\otimes k[\epsilon]\xrightarrow{d^i_{V^\bullet}+\epsilon v^i} V_{i+1}\otimes\mathcal{O}_X(-m_{i+1})\otimes k[\epsilon]\longrightarrow\cdots.$$

Consider the surjection $k[t]/(t^3)\to k[\epsilon]$; $t\mapsto\epsilon$ and the extension

$$d^i_{V^\bullet}+tv^i : V_i\otimes\mathcal{O}_X(-m_i)\otimes k[t]/\big(t^3\big)\longrightarrow V_{i+1}\otimes\mathcal{O}_X(-m_{i+1})\otimes k[t]/\big(t^3\big)$$

of the homomorphism $d^i_{V^\bullet}+\epsilon v^i : V_i\otimes\mathcal{O}_X(-m_i)\otimes k[\epsilon]\to V_{i+1}\otimes\mathcal{O}_X(-m_{i+1})\otimes k[\epsilon]$. Then the obstruction class $\omega(\tilde{V}^\bullet)$ for the lifting of $\tilde{V}^\bullet$ to a member of $\mathrm{Splcpx}_{X/k}(k[t]/(t^3))$ with respect to the surjection $k[t]/(t^3)\to k[\epsilon]$; $t\mapsto\epsilon$ is given by $[\{(d^{i+1}_{V^\bullet}+tv^{i+1})\circ(d^i_{V^\bullet}+tv^i)\}]\in(t^2)\otimes\mathrm{Ext}^2(V^\bullet,V^\bullet)$. However,

$$\begin{aligned}\big(d^{i+1}_{V^\bullet}+tv^{i+1}\big)\circ\big(d^i_{V^\bullet}+tv^i\big)&=d^{i+1}_{V^\bullet}\circ d^i_{V^\bullet}+t\big(d^{i+1}_{V^\bullet}\circ v^i+v^{i+1}\circ d^i_{V^\bullet}\big)+t^2v^{i+1}\circ v^i\\&=t^2v^{i+1}\circ v^i.\end{aligned}$$

Then $\alpha(p)(v,v)=v\circ v=[\{v^{i+1}\circ v^i\}]=\omega(\tilde{V}^\bullet)=0$ since $\mathrm{Splcpx}^{\text{ét}}_{X/k}$ is smooth over k.

Next we will see that α is nondegenerate. The canonical isomorphism

$$\mathbf{R}\mathcal{H}om\big(E^\bullet_U,E^\bullet_U\big)\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet\big(E^\bullet_U,E^\bullet_U\big)^\vee$$

induces the composite isomorphism by Grothendieck–Serre duality

$$\mathrm{Ext}^1\big(E^\bullet_U,E^\bullet_U\big)\xrightarrow{\sim}\mathrm{Ext}^1\big(\mathbf{R}\mathcal{H}om^\bullet\big(E^\bullet_U,E^\bullet_U\big),\mathcal{O}_{X_U}\big)\xrightarrow{\sim}\mathrm{Hom}\big(\mathrm{Ext}^1\big(E^\bullet_U,E^\bullet_U\big),\mathcal{O}_U\big),$$

which is just the homomorphism induced by α. Thus α is nondegenerate.

Finally we will show that α is d-closed. For an affine scheme U étale over $\coprod_i U_i$, take $u,v,w\in T_{\mathrm{Splcpx}^{\text{ét}}_{X/k}}(U)$. Let $E^\bullet\in\mathrm{Splcpx}_{X/k}(U)$ be the pullback of the universal family. We may assume that there exists a complex $V^\bullet$ of the form $V^i=V_i\otimes\mathcal{O}_{X_U}(-m_i)$ such that $V^\bullet$ is quasi-isomorphic to $E^\bullet$ and that V_i are vector spaces of finite dimension over k and m_i are integers. Take $u\in T_U(U)$. u can be regarded as a derivation $\mathcal{O}_U\to\mathcal{O}_U$ over $\mathcal{O}_U$, which is canonically extended to a derivation

$$D_u : V_i^\vee\otimes V_j\otimes\mathcal{O}_X(m_i-m_j)\otimes\mathcal{O}_U\longrightarrow V_i^\vee\otimes V_j\otimes\mathcal{O}_X(m_i-m_j)\otimes\mathcal{O}_U$$

for $i\leqslant j$. We have $d^{i+1}_{V^\bullet}\circ D_u(d^i_{V^\bullet})+D_u(d^{i+1}_{V^\bullet})\circ d^i_{V^\bullet}=0$ for any i. So we have $[\{D_u(d^i_{V^\bullet})\}]\in\mathrm{Ext}^1(V^\bullet,V^\bullet)$, which corresponds to u by the isomorphism $T_U(U)\xrightarrow{\sim}\mathrm{Ext}^1(V^\bullet,V^\bullet)$. Note that for $u,v\in T_U(U)$ we have

$$\alpha(u,v)=\big[\big\{D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big)\big\}\big]\in\mathrm{Ext}^2\big(V^\bullet,V^\bullet\big)\cong H^0(U,\mathcal{O}_U).$$

For $u,v,w\in T_U(U)$, we have

$$
\begin{aligned}
d\alpha(u,v,w) &= \big[\big\{D_u\big(\alpha(v,w)\big)+D_v\big(\alpha(w,u)\big)+D_w\big(\alpha(u,v)\big)+\alpha\big(w,[u,v]\big) \\
&\quad +\alpha\big([u,w],v\big)+\alpha\big(u,[v,w]\big)\big\}\big] \\
&= \big[\big\{D_u\big(D_v\big(d^{i+1}_{V^\bullet}\big)\circ D_w\big(d^i_{V^\bullet}\big)\big)+D_v\big(D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_u\big(d^i_{V^\bullet}\big)\big) \\
&\quad + D_w\big(D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big)\big)+D_w\big(d^{i+1}_{V^\bullet}\big)\circ(D_uD_v-D_vD_u)\big(d^i_{V^\bullet}\big) \\
&\quad +(D_uD_w-D_wD_u)\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big)+D_u\big(d^{i+1}_{V^\bullet}\big)\circ(D_vD_w-D_wD_v)\big(d^i_{V^\bullet}\big)\big\}\big] \\
&= \big[\big\{D_uD_v\big(d^{i+1}_{V^\bullet}\big)\circ D_w\big(d^i_{V^\bullet}\big)+D_v\big(d^{i+1}_{V^\bullet}\big)\circ D_uD_w\big(d^i_{V^\bullet}\big)+D_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_u\big(d^i_{V^\bullet}\big) \\
&\quad + D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_vD_u\big(d^i_{V^\bullet}\big)+D_wD_u\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big) \\
&\quad + D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_wD_v\big(d^i_{V^\bullet}\big)+D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_uD_v\big(d^i_{V^\bullet}\big) \\
&\quad - D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_vD_u\big(d^i_{V^\bullet}\big)+D_uD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big) \\
&\quad - D_wD_u\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big)+D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_vD_w\big(d^i_{V^\bullet}\big) \\
&\quad - D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_wD_v\big(d^i_{V^\bullet}\big)\big\}\big] \\
&= \big[\big\{D_uD_v\big(d^{i+1}_{V^\bullet}\big)\circ D_w\big(d^i_{V^\bullet}\big)+D_v\big(d^{i+1}_{V^\bullet}\big)\circ D_uD_w\big(d^i_{V^\bullet}\big)+D_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_u\big(d^i_{V^\bullet}\big) \\
&\quad + D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_uD_v\big(d^i_{V^\bullet}\big)+D_uD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big) \\
&\quad + D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_vD_w\big(d^i_{V^\bullet}\big)\big\}\big] \\
&= \big[\big\{D_uD_vD_w\big(d^{i+1}_{V^\bullet}\circ d^i_{V^\bullet}\big)\big\}-\big\{D_uD_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ d^i_{V^\bullet}+d^{i+1}_{V^\bullet}\circ D_uD_vD_w(d^i_{V^\bullet})\big\}\big] \\
&= \big[\big\{D_uD_vD_w(0)\big\}-d\big\{D_uD_vD_w\big(d^i_{V^\bullet}\big)\big\}\big] \\
&= 0.
\end{aligned}
$$

Here we used

$$
\begin{aligned}
D_uD_vD_w\big(d^{i+1}_{V^\bullet}\circ d^i_{V^\bullet}\big) &= D_u\big(D_v\big(D_w\big(d^{i+1}_{V^\bullet}\circ d^i_{V^\bullet}\big)\big)\big) \\
&= D_u\big(D_v\big(D_w\big(d^{i+1}_{V^\bullet}\big)\circ d^i_{V^\bullet}+d^{i+1}_{V^\bullet}\circ D_w\big(d^i_{V^\bullet}\big)\big)\big) \\
&= D_u\big(D_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ d^i_{V^\bullet}+D_w\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big) \\
&\quad + D_v\big(d^{i+1}_{V^\bullet}\big)\circ D_w\big(d^i_{V^\bullet}\big)+d^{i+1}_{V^\bullet}\circ D_vD_w\big(d^i_{V^\bullet}\big)\big) \\
&= D_uD_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ d^i_{V^\bullet}+D_vD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_u\big(d^i_{V^\bullet}\big) \\
&\quad + D_uD_w\big(d^{i+1}_{V^\bullet}\big)\circ D_v\big(d^i_{V^\bullet}\big)+D_w\big(d^{i+1}_{V^\bullet}\big)D_uD_v\big(d^i_{V^\bullet}\big) \\
&\quad + D_uD_v\big(d^{i+1}_{V^\bullet}\big)\circ D_w\big(d^i_{V^\bullet}\big)+D_v\big(d^{i+1}_{V^\bullet}\big)\circ D_uD_w\big(d^i_{V^\bullet}\big) \\
&\quad + D_u\big(d^{i+1}_{V^\bullet}\big)\circ D_vD_w\big(d^i_{V^\bullet}\big)+d^{i+1}_{V^\bullet}\circ D_uD_vD_w\big(d^i_{V^\bullet}\big).
\end{aligned}
$$

So α is a closed 2-form. □

Acknowledgments

The author would like to thank Professors Akira Ishii and Kōta Yoshioka for giving him the problem solved in this paper. The author would also like to thank Professor Fumiharu Kato for teaching him a fundamental concepts of algebraic spaces.

References

[1] A. Grothendieck, et al., Revêtements étales et Groupe Fondemental, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Lecture Notes in Math., vol. 224, Springer-Verlag, Heidelberg, 1971.
[2] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves, Aspects Math. 31 (1997).
[3] D. Huybrechts, R.P. Thomas, Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes, Math. Ann. 346 (2010) 545–569.
[4] M. Inaba, Toward a definition of moduli of complexes of coherent sheaves on a projective scheme, J. Math. Kyoto Univ. 42 (2) (2002) 317–329.
[5] M. Inaba, Moduli of stable objects in a triangulated category, J. Math. Soc. Japan 62 (2) (2010) 395–429; arXiv: math/0612078.
[6] M. Lieblich, Moduli of complexes on a proper morphism, J. Algebraic Geom. 15 (1) (2006) 175–206.
[7] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math. 77 (1984) 101–116.